Int. J. Solids Structures Vol. 29, No. 22, pp. 2689–2708, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00
© 1992 Pergamon Press Ltd

THE FREE VIBRATIONS OF INHOMOGENEOUS ELASTIC CYLINDERS AND SPHERES

P. R. HEYLIGER and A. JILANI
Department of Civil Engineering, Colorado State University, Fort Collins, CO 80523, U.S.A.

(*Received* 18 *November* 1991 ; *in revised form* 19 *March* 1992)

Abstract—The variational statement, governing equations and corresponding Ritz approximations are derived in Cartesian, cylindrical and spherical coordinates for the evaluation of the natural frequencies of free vibrations of elastic cylinders and spheres. The formulation can account for orthotropic material symmetry, and can be applied to either solid or hollow geometries. The approximating functions selected for the displacement components depend on the geometry and coordinate system used to describe the problem, and are a combination of power series, Fourier series and spherical harmonics. Representative examples are given in the various coordinate systems for both the cylinder and the sphere. In general, excellent agreement is found with results obtained by other methods.

1. INTRODUCTION

The study of free vibrations of homogeneous elastic bodies has been an important area of solid mechanics for well over a century. Initial studies of the vibrations of spheres were applied to analyse the oscillations of the earth. Knowledge of vibrations in cylinders can be applied to problems in beam and shell dynamics and the use of solid cylinders as underwater transducers or ultrastable microwave oscillators. Similar needs exist for single crystals for applications such as piezoelectric transducers and acoustic resonators.

On a much smaller scale, theories of vibration for many geometries can be used in the determination of elastic constants, which represent some of the most basic and useful properties of solid continua. This type of approach is particularly applicable to many advanced materials such as oxide superconductors, ceramics and composites. Not only are the materials often available only in these special shapes, but many traditional methods for evaluating the elastic constants are intractable for these solids. So-called solid resonance methods require techniques for accurately evaluating the natural frequencies of free vibration for solids of various shapes and constitutive laws.

The purpose of this work is to extend previous solutions to the three-dimensional equations of elasticity for the evaluation of free vibration frequencies using the Ritz method. Earlier works have focussed on homogeneous isotropic solids with the elastic constants and geometry defined in terms of rectangular Cartesian coordinates. The current approach generalizes this approach and allows for the additional consideration of a wide variety of new problems.

2. BACKGROUND

In the development of the theory of elasticity, the physics of small vibrations in elastic solids was of prominent importance. This was particularly true for solids of regular shape such as cylinders, spheres and parallelepipeds. Kelvin (1863) was the first to analyse this problem to study the rigidity of the earth. Later formulations for this class of problem were given by Lamb (1882), who formulated the problem in rectangular coordinates, and Chree (1889), who considered the same problem using spherical coordinates. In these cases, the sphere was assumed to be homogeneous and isotropic.

Numerous studies were subsequently completed for this type of problem because of intense interest in the oscillations of the earth [see, for example, Lapwood and Usami (1961), Ness *et al.* (1961), Pekeris *et al.* (1961) and Usami and Sato (1962a,b)]. Most of the interest in this subject was aimed at understanding the generation of earthquakes.

Although the earth is not a true sphere, but an ellipsoid, these works nevertheless provided a reasonably accurate prediction between numerical results and experimental observations. More recent studies on sphere vibrations have been completed regarding the orthogonality and normalization of torsional modes (Hosseini-Hashemi and Anderson, 1988) and spheres subjected to impulsive loading (Hosseini-Hashemi, 1986) and the evaluation of elastic constants (Mochizuki, 1988).

The analysis of free vibration of anisotropic spheres is complicated because the resonant conditions are highly dependent on the orientation of the material axes of the solid. In the case of isotropic materials, the analysis is greatly simplified and exact mathematical solutions are possible. To date, however, exact mathematical solutions for anisotropic spheres and other shapes do not exist.

The analysis of vibrating cylinders also has a rich history in the field of solid mechanics. This problem was first studied by Pochhammer (1876) and independently by Chree (1886). These solutions are exact for infinitely long isotropic rods that are stress free along their radial faces. Tables of natural frequencies and mode shapes of infinitely long solid circular cylinders have been compiled by Armenakas *et al.* (1969) using these theories. However, the Pochhammer–Chree theory cannot be applied to finite rods because the condition of stress-free ends cannot be imposed. Exact solutions for this case are difficult to obtain, and approximate methods must be used to study these geometries.

There have been numerous attempts made at obtaining approximate solutions for the free vibrations of finite isotropic rods. Bancroft (1941) was the first to numerically explore the governing equations of Pochhammer. His resulting values for various parameters related to longitudinal wave propagation agreed very well with experimental results obtained earlier by McMahon (1964). Later works studying the free vibrations of cylinders included the so-called three-mode theory (Mindlin and McNiven, 1960; McNiven and Perry, 1962), the series solution by Hutchinson (1972), the finite difference solution by McMahon (1970) and the finite element solutions by Gladwell and Tahbildar (1972) and Gladwell and Vijay (1975). Most of these methods gave frequencies that compared well with experimental observations.

Several studies have also appeared that consider the vibrations of finite anisotropic cylinders. Mörse (1954) extended the Pochhammer–Chree solutions to the case of an infinitely long cylinder composed of a material with hexagonal symmetry. Lusher and Hardy (1988) used Morse's approach to extend Hutchinson's solution in obtaining a series solution for a finite rod with hexagonal symmetry. Only the axisymmetric modes were considered in this work, which also included experimental observations made on sapphire. More recently, the Ritz method has been used to find estimates for the axisymmetric free vibrations of finite anisotropic cylinders (Heyliger, 1991).

The present work uses the Ritz method to find the natural frequencies of free vibration of anisotropic cylinders and spheres as formulated in cylindrical, spherical and Cartesian coordinates. This approach is based on the work of Eer Nisse (1987), Holland (1968), Demarest (1969) and Ohno (1976) and the recent work of Mochizuki (1988), Heyliger *et al.* (1989), Migliori *et al.* (1990) and Visscher *et al.* (1991). The present approach differs from these approaches in the coordinate systems, constitutive relations, and form of approximating functions that are used. This allows for the consideration of a wide variety of new problems that for example consider spatial dependence of the elastic stiffnesses (e.g. a layered shell) or solids where the principal material directions are not aligned with the directions of a rectangular Cartesian coordinate system (e.g. a filament-wound cylindrical composite shell).

3. GOVERNING EQUATIONS

In this section, the equations of motion and their corresponding weak forms are derived, and the Ritz approximations to these equations for the orthotropic cylinder and sphere are constructed in several different coordinate systems. The representative equations are derived for the orthotropic cylinder and sphere in cylindrical and spherical coordinates, respectively, and are also derived in Cartesian coordinates. Each of these different systems

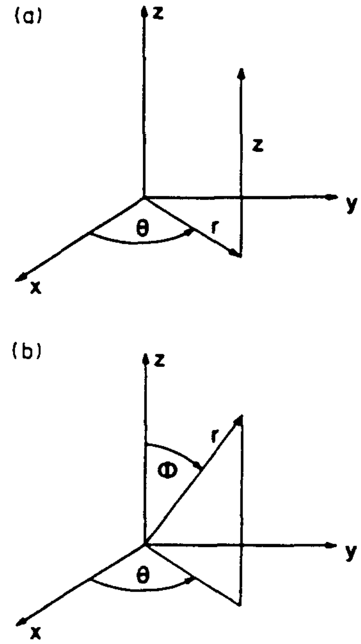

Fig. 1. Coordinate systems: (a) cylindrical coordinates; (b) spherical coordinates.

is represented in Fig. 1. The Ritz approximations are developed using basis functions that match the coordinate system under consideration. As shown in the sequel, use of different coordinate systems allows a more natural representation of a problem based on the nature of the elastic stiffnesses and the geometry of the solid.

In the following discussion, the generic coordinate system x_i is introduced, with the appropriate directions depending on the specific coordinate system. The general constitutive relation that will be considered here can be expressed in the form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}. \quad (1)$$

Here σ_{ij} represent the components of stress, C_{ij} are the components of the elastic stiffness tensor and ε and γ are the normal and shear strain components, respectively. The representative directions defined by the subscripts will be identified for each case in the sequel. Clearly the derivation that follows can be easily modified for more complex constitutive relations. Hamilton's principle provides the starting point for each of the subsequent derivations. This can be written as

$$0 = -\int_0^t \int_V \{\sigma_1\delta\varepsilon_1 + \sigma_2\delta\varepsilon_2 + \sigma_3\delta\varepsilon_3 + \sigma_4\delta\varepsilon_4 + \sigma_5\delta\varepsilon_5 + \sigma_6\delta\varepsilon_6\}\,\mathrm{d}V\,\mathrm{d}t + \tfrac{1}{2}\delta\int_0^t \int_V \rho(\dot{U}^2 + \dot{V}^2 + \dot{W}^2)\,\mathrm{d}V\,\mathrm{d}t. \quad (2)$$

Here, V is the volume of the solid, $\dot{u} = \partial u/\partial t$, t represents time, ρ is the density of the material, δ is the variational operator, U, V and W are the displacement components in the three coordinate directions, and the conventional notation ($\sigma_{11} = \sigma_1$, $\sigma_{23} = \sigma_4$, etc.) has been used. For the problem of free vibration, there are no applied external loads and each face of the solid is assumed to be stress free.

The orthotropic sphere : cylindrical coordinates

For the orthotropic cylinder, the displacement components are identified in the radial, tangential and axial directions. Hence $x_1 = r$, $x_2 = \theta$ and $x_3 = z$ as shown in Fig. 1a and $U_1 = U_r = U(r,\theta,z)$, $U_2 = U_\theta = V(r,\theta,z)$ and $U_3 = U_z = W(r,\theta,z)$. Formulating the equations of motion in this form is convenient not only for cylindrical geometries but also for materials such as wood and filament-wound composites, where the principal material directions are most naturally defined in the radial, circumferential and axial directions. The stress–strain relations in cylindrical coordinates are given as (Fung, 1965)

$$\varepsilon_{rr} = \frac{\partial U}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r}, \quad \varepsilon_{zz} = \frac{\partial W}{\partial z}, \tag{3}$$

$$\gamma_{\theta z} = \frac{\partial V}{\partial z} + \frac{1}{r}\frac{\partial W}{\partial \theta}, \quad \gamma_{rz} = \frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}, \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r}. \tag{4}$$

To aid in the calculations, the displacement variables are nondimensionalized by defining

$$U^* = \frac{U}{L_r}, \quad V^* = V, \quad W^* = \frac{W}{L_z}, \tag{5}$$

where L_r and L_z represent the radius and half-height of the cylinder, respectively. This procedure is used only for computational efficiency by allowing all cylinders to be mapped into a parent cylinder to minimize computations. For simplicity, it is assumed that this process has been completed, and the asterisks are dropped.

Substitution of the strain–displacement and stress–strain relations into Hamilton's principle, using the assumption of periodic motion, and performing the appropriate operations yields the variational form of the governing equations as

$$\begin{aligned}
0 = \int_V \Bigg[& \frac{C_{11}}{L_r^2}\frac{\partial U}{\partial r}\frac{\partial \delta U}{\partial r} + \frac{1}{r}\frac{C_{12}}{L_r^2}\frac{\partial U}{\partial r}\delta U + \frac{1}{r}\frac{C_{12}}{L_r^2}\frac{\partial \delta U}{\partial r}U + \frac{1}{r}\frac{C_{12}}{L_r^2}\frac{\partial V}{\partial \theta}\frac{\partial \delta U}{\partial r} \\
& + \frac{1}{r}\frac{C_{12}}{L_r^2}\frac{\partial \delta V}{\partial \theta}\frac{\partial U}{\partial r} + \frac{C_{13}}{L_r L_z}\frac{\partial U}{\partial r}\frac{\partial \delta W}{\partial z} + \frac{C_{13}}{L_r L_z}\frac{\partial \delta U}{\partial r}\frac{\partial W}{\partial z} + \frac{1}{r}\frac{C_{23}}{L_r L_z}\frac{\partial W}{\partial z}\frac{\partial \delta V}{\partial \theta} \\
& + \frac{1}{r}\frac{C_{23}}{L_r L_z}\frac{\partial \delta W}{\partial z}\frac{\partial V}{\partial \theta} + \frac{1}{r}\frac{C_{23}}{L_r L_z}\frac{\partial W}{\partial z}\delta U + \frac{1}{r}\frac{C_{23}}{L_r L_z}\frac{\partial \delta W}{\partial z}U + \frac{1}{r^2}\frac{C_{22}}{L_r^2}\frac{\partial V}{\partial \theta}\frac{\partial \delta V}{\partial \theta} \\
& + \frac{1}{r^2}\frac{C_{22}}{L_r^2}\frac{\partial V}{\partial \theta}\delta U + \frac{1}{r^2}\frac{C_{22}}{L_r^2}\frac{\partial \delta V}{\partial \theta}U + \frac{1}{r^2}\frac{C_{22}}{L_r^2}U\delta U + \frac{C_{33}}{L_r^2}\frac{\partial W}{\partial z}\frac{\partial \delta W}{\partial z} \\
& + C_{44}\left(\frac{1}{L_z}\frac{\partial V}{\partial z} + \frac{1}{r}\frac{1}{L_r}\frac{\partial W}{\partial \theta}\right)\left(\frac{1}{L_z}\frac{\partial \delta V}{\partial z} + \frac{1}{r}\frac{1}{L_r}\frac{\partial \delta W}{\partial \theta}\right) + C_{55}\left(\frac{1}{L_r}\frac{\partial W}{\partial r} + \frac{1}{L_z}\frac{\partial U}{\partial z}\right) \\
& \times \left(\frac{1}{L_r}\frac{\partial \delta W}{\partial r} + \frac{1}{L_z}\frac{\partial \delta U}{\partial z}\right) + C_{66}\left(\frac{1}{r}\frac{1}{L_r}\frac{\partial U}{\partial \theta} + \frac{1}{L_r}\frac{\partial U}{\partial r} - \frac{1}{L_r}\frac{V}{r}\right) \\
& \times \left(\frac{1}{r}\frac{1}{L_r}\frac{\partial \delta U}{\partial \theta} + \frac{1}{L_r}\frac{\partial \delta V}{\partial r} - \frac{1}{L_r}\frac{\delta V}{r}\right) - \rho\ddot{U} - \rho\ddot{V} - \rho\ddot{W}\Bigg]\mathrm{d}V.
\end{aligned} \tag{6}$$

This equation forms the basis for numerical approximations using the Ritz method. Integrating by parts and separating the coefficients of the variations in U, V and W yields the equations of motion. These are given in Appendix A.

The orthotropic sphere : spherical coordinates

In spherical coordinates, the displacement components are defined in the radial ($x_1 = r$), azimuthal ($x_2 = \phi$), and circumferential ($x_3 = \theta$) directions as shown in Fig. 1b as $U_1 = U_r = U(r, \phi, \theta)$, $U_2 = U_\phi = V(r, \phi, \theta)$, $U_3 = U_\theta = W(r, \phi, \theta)$. The stress–strain relations can be written as

$$\varepsilon_{rr} = \frac{\partial U}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U}{r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r\sin\phi}\frac{\partial W}{\partial \theta} + \frac{U}{r} + \frac{\cot\phi}{r}V,$$

$$\gamma_{r\phi} = \frac{1}{r}\frac{\partial U}{\partial \phi} + \frac{\partial V}{\partial r} - \frac{V}{r}, \quad \gamma_{\phi\theta} = \frac{1}{r\sin\phi}\frac{\partial V}{\partial \theta} + \frac{1}{r}\frac{\partial W}{\partial \phi} - \frac{\cot\phi}{r}W,$$

$$\gamma_{r\theta} = \frac{1}{r\sin\phi}\frac{\partial U}{\partial \theta} + \frac{\partial W}{\partial r} - \frac{W}{r}. \tag{7}$$

The corresponding weak form is given by

$$\begin{aligned}
0 = \int_V \Bigg\{ & \left[C_{11}\frac{\partial U}{\partial r} + C_{12}\left(\frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U}{r}\right) + C_{13}\left(\frac{1}{r\sin\phi}\frac{\partial W}{\partial \theta} + \frac{U}{r} + \frac{\cot\phi}{r}V\right)\right] \\
& \times \frac{\partial \delta U}{\partial r} + \left[C_{12}\frac{\partial U}{\partial r} + C_{22}\left(\frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U}{r}\right) + C_{23}\left(\frac{1}{r\sin\phi}\frac{\partial W}{\partial \theta} + \frac{U}{r} + \frac{\cot\phi}{r}V\right)\right] \\
& \times \left(\frac{1}{r}\frac{\partial \delta V}{\partial \phi} + \frac{\delta U}{r}\right)\left[C_{13}\frac{\partial U}{\partial r} + C_{23}\left(\frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U}{r}\right) + C_{33}\left(\frac{1}{r\sin\phi}\frac{\partial W}{\partial \theta} + \frac{U}{r} + \frac{\cot\phi}{r}V\right)\right] \\
& \times \left(\frac{1}{r\sin\phi}\frac{\partial \delta W}{\partial \theta} + \frac{\delta U}{r} + \frac{\cot\phi}{r}\delta V\right) + C_{66}\left(\frac{1}{r}\frac{\partial U}{\partial \phi} + \frac{\partial V}{\partial r} - \frac{V}{r}\right)\left(\frac{1}{r}\frac{\partial \delta U}{\partial \phi} + \frac{\partial \delta V}{\partial r} - \frac{\delta V}{r}\right) \\
& + C_{55}\left(\frac{1}{r\sin\phi}\frac{\partial V}{\partial \theta} + \frac{1}{r}\frac{\partial W}{\partial \phi} - \frac{\cot\phi}{r}W\right)\left(\frac{1}{r\sin\phi}\frac{\partial \delta V}{\partial \theta} + \frac{1}{r}\frac{\partial \delta W}{\partial \phi} - \frac{\cot\phi}{r}\partial W\right) \\
& + C_{44}\left(\frac{1}{r\sin\phi}\frac{\partial U}{\partial \theta} + \frac{\partial W}{\partial r} - \frac{W}{r}\right)\left(\frac{1}{r\sin\phi}\frac{\partial \delta U}{\partial \theta} + \frac{\partial \delta W}{\partial r} - \frac{\delta W}{r}\right) \\
& - \rho \ddot{U}_r - \rho \ddot{U}_\phi - \rho \ddot{U}_\theta \Bigg\} \mathrm{d}V.
\end{aligned} \tag{8}$$

Although not shown, the radial displacement component can be nondimensionalized so that all calculations are performed on the unit sphere. The corresponding equations of motion are given in Appendix A.

The orthotropic solid: Cartesian coordinates

The problem of the vibrating solid can also be formulated in terms of rectangular Cartesian coordinates. This approach has the advantage that the modes of vibration can be separated according to certain geometric and material symmetries, which drastically reduce the size of the eigenvalue problem that results from these formulations (Demarest, 1969). In this case the displacement components are expressed in terms of the coordinates ($x_1 = x, x_2 = y, x_3 = z$) with the corresponding displacement components $U_1 = U(x, y, z)$, $U_2 = V(x, y, z)$ and $U_3 = W(x, y, z)$. The strain–displacement relations are given as

$$\varepsilon_{11} = \frac{\partial U}{\partial x}, \quad \varepsilon_{22} = \frac{\partial V}{\partial y}, \quad \varepsilon_{33} = \frac{\partial W}{\partial z},$$

$$\gamma_{23} = \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}, \quad \gamma_{13} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}, \quad \gamma_{12} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}. \tag{9}$$

Substitution of these equations and the stress–strain relations into eqn (2) gives the weak form as

$$\delta U = \int_V \left[\left(C_{11}\frac{\partial U}{\partial x} + C_{12}\frac{\partial V}{\partial y} + C_{13}\frac{\partial W}{\partial z} \right)\frac{\partial \delta U}{\partial x} + \left(C_{12}\frac{\partial U}{\partial x} + C_{22}\frac{\partial V}{\partial y} + C_{23}\frac{\partial W}{\partial z} \right)\frac{\partial \delta V}{\partial y} \right.$$
$$+ \left(C_{13}\frac{\partial U}{\partial x} + C_{23}\frac{\partial V}{\partial y} + C_{33}\frac{\partial W}{\partial z} \right)\frac{\partial \delta W}{\partial z} + C_{44}\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right)\left(\frac{\partial \delta V}{\partial z} + \frac{\partial \delta W}{\partial y}\right)$$
$$+ C_{55}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)\left(\frac{\partial \delta U}{\partial z} + \frac{\partial \delta W}{\partial x}\right) + C_{66}\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right)\left(\frac{\partial \delta U}{\partial y} + \frac{\partial \delta V}{\partial x}\right)$$
$$\left. -\rho\omega^2(U\delta U + V\delta V + W\delta W) \right] \mathrm{d}V. \tag{10}$$

Ritz approximations

Rather than directly solving the governing differential equations, the Ritz method seeks approximate solutions to the weak forms as given in eqns (6), (8) and (10). This is accomplished by approximating the displacement components U, V and W using finite linear combinations of the form [see Reddy (1984)]:

$$U(x_1,x_2,x_3) = \phi_0^u(x_1,x_2,x_3) + \sum_{i=1}^{n} a_i\phi_i^u(x_1,x_2,x_3), \quad \delta U = \phi_i^u(x_1,x_2,x_3),$$
$$V(x_1,x_2,x_3) = \phi_0^v(x_1,x_2,x_3) + \sum_{i=1}^{n} b_i\phi_i^v(x_1,x_2,x_3), \quad \delta V = \phi_i^v(x_1,x_2,x_3),$$
$$W(x_1,x_2,x_3) = \phi_0^w(x_1,x_2,x_3) + \sum_{i=1}^{n} d_i\phi_i^w(x_1,x_2,x_3), \quad \delta W = \phi_i^w(x_1,x_2,x_3). \tag{11}$$

Here ϕ_i^u etc. are known functions of position, n represents the number of terms in the approximation for the displacement components, and a_i, b_i and d_i are constants that are determined by requiring that each of the variational statements hold for arbitrary variations of U, V and W. This latter requirement is equivalent to the weak forms holding for arbitrary variations of a, b and d.

The selection of the approximating functions ϕ is somewhat arbitrary as long as several requirements are met in order for the Ritz approximations to converge to the true solution. First, the functions ϕ_0 should satisfy the actual form of the specified essential boundary conditions. For the problem of free vibration, the boundary conditions are all of the natural type in that all faces of the solid are stress free. In the Ritz method, the natural boundary conditions are contained in the variational statement of the problem. Hence there is no need to explicitly satisfy these conditions. For these reasons, the ϕ_0 terms are set equal to zero. The remaining functions in the summation must meet the requirement of continuity as required by the variational statement, they must satisfy the homogeneous form of the essential boundary conditions, and they must be linearly independent.

A number of different functions can be selected which meet these requirements. In this work, a combination of power series, Fourier series and spherical harmonics are used to approximate the displacements. The actual selection of these functions depends on the class of vibration and the geometry of the problem being solved. In cylindrical coordinates, Fourier series are used in the circumferential direction, with power series used in the radial and axial directions. In spherical coordinates, power series in the radial directions are combined with spherical harmonics. Formulation of the problem in Cartesian coordinates leads to approximations for each displacement component in the form of power series in the coordinate directions (Heyliger, 1991).

Substitution of the approximate displacements and their variations into the weak forms and collecting terms allows for writing the final equation in matrix form as

$$\left[\begin{bmatrix} K^{11} \\ K^{21} \\ K^{31} \end{bmatrix} \begin{bmatrix} K^{12} \\ K^{22} \\ K^{32} \end{bmatrix} \begin{bmatrix} K^{13} \\ K^{23} \\ K^{33} \end{bmatrix}\right] \begin{Bmatrix} \{a\} \\ \{b\} \\ \{d\} \end{Bmatrix} - \rho\omega^2 \begin{bmatrix} [M^{11}] & [0] & [0] \\ [0] & [M^{22}] & [0] \\ [0] & [0] & [M^{33}] \end{bmatrix} \begin{Bmatrix} \{a\} \\ \{b\} \\ \{d\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix}. \tag{12}$$

The nature of $[K]$ and $[M]$ is a function of the particular formulation used. Depending on the problem, a combination of analytic and numerical integration techniques are used to evaluate the individual elements of these matrices. Numerical integration is particularly useful when the elastic stiffnesses vary as a function of position and the resulting integrals cannot generally be evaluated in closed form. The explicit forms of the coefficient matrices for the different coordinate systems are given in Appendix A. By using any conventional solution technique, this system of equations can be solved for the natural frequencies of the cylinder.

4. NUMERICAL RESULTS

In this section, results are presented from the application of the Ritz method to representative example problems corresponding to the formulations in the previous section. These problems represent the isotropic and anisotropic cylinder and sphere as formulated in the different coordinate systems accounting for the possibility of material inhomogeneity. The primary quantities of interest are the natural frequencies of free vibration. Although calculating and presenting the mode shapes involves little additional effort, results of this type are referred to but not shown in this paper.

An additional measure of accuracy of the present technique is how well the condition of traction-free surfaces is satisfied. In the cases of the isotropic sphere, the hollow isotropic cylinder, and the cross-ply shell, the stresses were integrated over all faces of the solid to assess the level of satisfaction of traction free surfaces. The resulting values were without exception infinitesimally small.

The solid isotropic sphere

The isotropic sphere is useful to examine, primarily because it is a problem for which exact solutions are available. In this section, results from the vibrations of the isotropic sphere as formulated in spherical coordinates are presented. Torsional and spheroidal vibrations of both hollow and solid spheres are considered.

For torsional vibrations there is no change in the volume of the sphere. This implies that the radial displacements are zero, the dilatation is zero, and the displacements are directed at right angles to radial lines from the center of the sphere. One form of the assumed displacements as represented by the approximating functions are those that coincide with the eigenfunctions derived from the exact solution for the isotropic sphere (see Appendix B). The displacement components in this case have the form

$$U_r = U(r, \theta, \phi) = 0, \tag{13}$$

$$U_\theta = V(r, \theta, \phi) = f_\theta(r) \frac{m}{\sin\theta} P_n^m \cos\theta)^{\sin m\phi}_{-\cos m\phi}, \tag{14}$$

$$U_\phi = W(r, \theta, \phi) = f_\phi(r) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta)^{\cos m\phi}_{\sin m\phi}. \tag{15}$$

Because the spherical harmonics are selected to approximate the displacements, the behavior in the θ and ϕ directions can be represented exactly and the only unknown functions are those in the radial direction. Hence the only terms that are truly approximate are those used to represent $f_\theta(r)$ and $f_\phi(r)$ in eqns (14) and (15) and the problem is essentially one-dimensional. In the radial direction, a power series in r is used to approximate these two functions.

Because of the physics of torsional vibrations, the resulting frequencies do not depend on the Poisson ratio. The nondimensional frequencies calculated using this approach are given in Table 1 as a function of the number of terms used in the power series. The results are compared with the exact solution, and the agreement is very good. The accuracy lessens somewhat with the higher modes, which is an expected property of the Ritz method. However, because the lower modes are of most interest, the accuracy for these frequencies is well within reasonable limits.

Vibrations that involve a change in shape of the sphere are referred to as spheroidal vibrations. These involve both transverse and radial motion at all points of the sphere, and correspondingly there is now a dependence of the frequencies on the Poisson ratio. For spheroidal vibrations, the approximation functions are selected such that the displacement components have a form similar to those of the analytic solution. These can be expressed as

$$U_r = U(r,\theta,\phi) = f_r(r)P_n^m(\cos\theta)_{\cos m\phi}^{\sin m\phi}, \tag{16}$$

$$U_\theta = V(r,\theta,\phi) = f_\theta(r)\frac{\mathrm{d}}{\mathrm{d}\theta}P_n^m(\cos\theta)_{\sin m\phi}^{\cos m\phi} \tag{17}$$

$$U_\phi = W(r,\theta,\phi) = f_\phi(r)\frac{m}{\sin\theta}P_n^m(\cos\theta)_{\cos m\phi}^{\sin m\phi}. \tag{18}$$

The resulting nondimensional frequencies are given in Table 2 for the case of $\nu = 0.3$. As before, the values are given as a function of the order used for the power series in the radial direction, with excellent accuracy being obtained for a relatively low number of terms. The results for the higher modes are somewhat more accurate than the higher frequencies for torsional vibrations.

Table 1. Nondimensional frequencies for torsional modes of an isotropic sphere

(a) $n = 1$

Mode	Terms of power series in radial direction: 3	4	5	6	7	Exact
1	0.0	0.0	0.0	0.0	0.0	None
2	6.687	5.781	5.774	5.763	5.763	5.763
3	12.254	12.038	9.198	9.186	9.096	9.095
4	—	18.699	18.496	12.666	12.651	12.323

(b) $n = 2$

Mode	Terms of power series in radial direction: 2	3	4	5	Exact
1	2.526	2.519	2.501	2.501	2.501
2	10.014	8.094	7.404	7.155	7.136
3	—	17.411	13.688	11.280	10.514
4	—	—	26.675	20.524	16.983

(c) $n = 3$

Mode	Order of terms of power series in radial direction: 0	1	2	3	4	5	Exact
1	5.745	3.955	3.905	3.869	3.865	3.865	3.865
2	—	13.380	9.363	9.120	8.470	8.468	8.444
3	—	—	23.283	14.893	13.775	12.023	11.881
4	—	—	—	35.889	21.653	18.948	15.175

Table 2. Nondimensional frequencies for spheroidal modes of an isotropic sphere

$n = 0$

	Order of terms in power series						
Mode	0	1	2	3	4	5	Exact
1	4.8990	4.7178	4.4414	4.4403	4.4400	4.4400	4.440
2	—	12.718	12.625	10.526	10.523	10.494	10.494
3	—	—	21.929	21.849	16.269	16.265	16.073
4	—	—	—	33.108	33.036	22.215	21.579

(b) $n = 1$

	Order of terms in power series							
Mode	0	1	2	3	4	5	6	Exact
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	6.7081	3.5122	3.4826	3.4247	3.3246	3.4245	3.4245	3.424
3	—	7.9897	6.9877	6.8910	6.7747	6.7732	6.7712	6.771
4	—	15.615	9.1424	8.0658	7.7896	7.7518	7.7458	7.744
5	—	—	26.851	13.3178	11.793	10.872	10.746	10.695

(c) $n = 2$

	Order of terms in power series							
Mode	0	1	2	3	4	5	6	Exact
1	3.0795	2.9593	2.6414	2.6406	2.640	2.640	2.640	2.640
2	9.1388	5.4448	5.0952	4.8782	4.8667	4.8653	4.8653	4.865
3	—	10.437	10.328	8.8423	8.4012	8.3520	8.3304	8.329
4	—	20.182	11.286	11.257	9.8448	9.8356	9.7816	9.780
5	—	—	18.822	17.230	15.0428	12.444	12.405	12.157

Although not shown, the shapes for most of these modes were plotted and compared with the exact mode shapes such as those presented by Hosseini-Hashemi and Anderson (1988). The comparison plots are virtually indistinguishable.

The solid isotropic cylinder: circumferential vibrations

The circumferential vibrations of the isotropic cylinder were considered as formulated in cylindrical coordinates. The primary focus for these geometries was the frequency for the circumferential wave numbers $n = 1$ and $n = 2$. These cases were selected because comparative finite element solutions for this class of vibration have been tabulated by Gladwell and Vijay (1975). The case of $n = 0$ corresponds to axisymmetric vibration and has already been considered using displacements of the type described in eqn (11) with $V = 0$ (Heyliger, 1991). These results are not repeated here.

The natural frequencies for the solid isotropic cylinder are presented in terms of the dimensionless frequency Ω, defined as $\omega a/c$, where $c = \sqrt{(G/\rho)}$, G represents the shear modulus, a is the mean radius, and ρ is the material density. The Poisson ratio is taken as 0.3 for the results in this section. Tables 3 and 4 show the convergence of the frequency parameter for the first eight circumferential modes as a function of the order of the approximation in the radial and axial directions. The geometry of the cylinders is expressed in terms of the shape parameter L, defined as the ratio between the cylinder length $2L$, and the mean radius a. Table 3 is representative of a short disc, with $L = 2$. Table 4 shows the converged frequency parameters for a range of values of the shape parameter L. The results agree quite well with the finite element results of Gladwell and Vijay (1975). In all cases, the present results are lower than those computed using finite elements. As the Ritz method converges to the exact solution from above, the present results can be expected to be more accurate than the finite element results.

The solid isotropic cylinder: general vibrations

An application of the solution to the equations of motion for a solid isotropic cylinder in both cylindrical and Cartesian coordinates is considered next. Specifically, the complete vibrations of a solid isotropic cylinder are examined in which all classes of vibrations are considered. This requires keeping all terms in the expansions for each displacement

Table 3. Convergence of frequency parameter ω^* for circumferential vibrations of a solid isotropic cylinder

(a) Circumferential wave number = 1

Mode	Order of terms 1	2	3	4	5	6	7	FEM
1	1.4170	1.3872	1.3526	1.3524	1.3523	1.3523	1.3523	1.3578
2	1.9763	1.6703	1.5522	1.5446	1.5440	1.5440	1.5440	1.5570
3	2.2613	2.1948	2.0327	2.0277	2.0264	2.0264	2.0264	2.0330
4	2.7363	2.5009	2.3256	2.2991	2.2976	2.2973	2.2973	2.3054
5	3.9844	2.7855	2.3450	2.3065	2.2997	2.2973	2.2973	2.3596
6	4.2478	2.8579	2.5230	2.4205	2.4191	2.4181	2.4181	2.4686
7	5.2601	3.7473	3.3125	3.2168	3.1957	3.1935	3.1932	3.2751
8	6.0621	3.9872	3.4681	3.2319	3.2251	3.2195	3.2194	3.2794

(b) Circumferential wave number = 2

Mode	Orders of terms 1	2	3	4	5	6	7	8	FEM
1	0.9494	0.8231	0.7968	0.7957	0.7957	0.7956	0.7956	0.7956	0.8016
2	1.3941	1.1758	1.1731	1.1725	1.1725	1.1725	1.1725	1.1725	1.1843
3	2.5346	2.0726	2.0009	1.9837	1.9833	1.9832	1.9832	1.9832	1.9934
4	2.6367	2.2029	2.0373	2.0220	2.0201	2.0200	2.0200	2.0200	2.0438
5	3.5581	2.7368	2.5105	2.4381	2.4343	2.4336	2.4336	2.4336	2.4620
6	3.8508	2.8550	2.6457	2.6241	2.6215	2.6213	2.6213	2.6213	2.6346
7	6.7965	3.5630	2.9433	2.8161	2.8124	2.8114	2.8114	2.8113	2.8948
8	6.9428	3.6760	3.2305	2.9488	2.9293	2.9252	2.9250	2.9249	3.0394
9	9.6965	4.5594	4.1379	3.7299	3.6953	3.6778	3.6766	3.6764	3.7761
10	10.1988	4.8603	4.2768	3.8532	3.8124	3.7979	3.7972	3.7970	3.9820

component rather than separating them according to the circumferential wave number. As discussed earlier, grouping the approximating functions as formulated in Cartesian coordinates in an appropriate fashion allows a reduction in size of the corresponding eigenvalue problem, making the solution procedure much more efficient. In both cases, terms in the approximation up to and including sixth order in eqn (11) are used for the Cartesian formulation and fourth order using cylindrical coordinates.

A cylinder is considered with unit density and radius and a height of 2.0. The shear modulus is also taken as unity and a Poisson ratio of 0.3 is specified. The resulting natural

Table 4. Variation of frequency parameter ω^* for solid isotropic cylinder with varying height

(a) Circumferential wave number = 1

Mode	$L = 2$ Present	FEM	$L = 4$ Present	FEM	$L = 6$ Present	FEM	$L = 8$ Present	FEM	$L = 10$ Present	FEM
1	1.3523	1.3578	0.9969	0.9980	0.5880	0.5886	0.3868	0.3871	0.2731	0.2733
2	1.5440	1.5570	1.0755	1.0792	0.8868	0.8891	0.7156	0.7169	0.5565	0.5573
3	2.0264	2.0330	1.4008	1.4072	1.2667	1.2711	1.0456	1.0484	0.8487	0.8506
4	2.2973	2.3054	1.5407	1.5453	1.2950	1.2988	1.0636	1.0670	0.9660	0.9690
5	2.2973	2.3596	1.7000	1.7137	1.4441	1.4513	1.3633	1.3682	1.1906	1.1954
6	2.4181	2.4686	1.9820	2.0025	1.4782	1.4844	1.3900	1.3961	1.2023	1.2061
7	3.1932	3.2751	2.0583	2.0623	1.6705	1.6756	1.4135	1.4200	1.3941	1.4000
8	3.2194	3.2794	2.2887	2.3185	1.8432	1.8567	1.5790	1.5875	1.4196	1.4268

(b) Circumferential wave number = 2

Mode	$L = 2$ Present	FEM	$L = 4$ Present	FEM	$L = 6$ Present	FEM	$L = 8$ Present	FEM	$L = 10$ Present	FEM
1	0.7956	0.8016	0.9892	0.9963	1.0533	1.0611	1.0589	1.0663	1.0603	1.0680
2	1.1725	1.1843	1.1690	1.1802	1.0814	1.0885	1.0729	1.0806	1.0717	1.0792
3	1.9832	1.9934	1.2439	1.2513	1.1751	1.1871	1.1743	1.1862	1.1740	1.1859
4	2.0200	2.0438	1.5961	1.6092	1.3464	1.3569	1.2177	1.2283	1.1766	1.1876
5	2.4336	2.4620	2.0190	2.0311	1.6805	1.6945	1.4290	1.4405	1.2892	1.2999
6	2.6213	2.6346	2.0349	2.0562	1.7033	1.7146	1.5887	1.5999	1.4605	1.4729
7	2.8113	2.8948	2.2286	2.2585	2.0364	2.0584	1.7560	1.7689	1.6176	1.6270
8	2.9249	3.0394	2.3520	2.3841	2.0397	2.0584	1.8011	1.8166	1.6439	1.6587
9	3.6764	3.7761	2.5675	2.5845	2.1808	2.2082	2.0397	2.0564	1.8237	1.8359
10	3.7970	3.9820	2.6987	2.7742	2.3307	2.3557	2.0466	2.0677	1.8499	1.8682

Table 5. Frequencies of solid isotropic cylinder as calculated in cylindrical and Cartesian coordinates

Mode	Cylindrical	Cartesian	Group
1	0.2500	0.2500	EV
2	0.3149	0.3149	EV
3	0.3149	0.3149	EZ
4	0.3173	0.3175	EY
5	0.3173	0.3175	EX
6	0.3424	0.3424	OX
7	0.3424	0.3424	OY
8	0.3702	0.3702	OD
9	0.3721	0.3721	OD
10	0.3721	0.3721	OZ
11	0.3959	0.3960	OZ
12	0.3959	0.3960	OD
13	0.4459	0.4463	EY
14	0.4459	0.4463	EX
15	0.4571	0.4579	EZ
16	0.4882	0.4883	OD
17	0.4904	0.4906	OX
18	0.4904	0.4906	OY
19	0.5000	0.5006	OZ

frequencies are given in Table 5. There is excellent agreement between the two formulations. Each approach yields the six zero eigenvalues corresponding to the rigid body modes, which are not included in the table. Also shown in the results are the corresponding group of each mode identified by the nomenclature of Ohno (1976) and others. Such a grouping infers that the vibration is of a specific type such as torsional, breathing and bending modes. The exact first nonzero frequency is equal to 0.25 (Landau *et al.*, 1986) and the exact 19th frequency is equal to 0.5, which provides some basis of comparison for the results.

The hollow isotropic sphere

It is a simple modification to evaluate the natural frequencies of hollow cylinders and spheres using any of the formulations. By mapping the thickness of the solid into the region of integration, a wide range of wall thicknesses can be represented. Table 6 shows the dimensionless frequencies for both torsional and spheroidal vibrations for a sphere with a wall thickness to radius ratio of 0.6. The results are compared with the analytic solution of Hossieni-Hashemi (1986) using terms up to seventh order in the approximation as formulated in spherical coordinates.

Table 6. Dimensionless frequencies for hollow isotropic sphere

(a) Torsional vibrations

		Mode			
N	Theory	1	2	3	4
1	Present	6.357	11.141	16.340	22.030
	Analytic	6.357	11.141	16.171	21.296
2	Present	2.475	7.237	11.639	16.515
	Analytic	2.475	7.237	11.639	16.499
3	Present	3.850	8.358	12.409	17.221
	Analytic	3.850	8.358	12.363	16.986

(b) Spheroidal vibrations ($\nu = 0.29$)

		Mode			
N	Theory	1	2	3	4
0	Present	4.057	10.751	19.793	29.270
	Analytic	4.070	10.781	19.855	29.322
1	Present	3.756	7.857	9.611	12.677
	Analytic	3.756	7.857	9.611	12.678
2	Present	2.165	4.827	9.288	14.089
	Analytic	2.165	4.832	9.298	14.115

Table 7. Nondimensional frequencies of a hollow isotropic cylinder

(a) Circumferential wave number = 0

Mode	$L = 2$		$L = 10$	
	Present	FEM	Present	FEM
1	1.4272	1.4278	0.5030	0.5030
2	1.7163	1.7168	0.9792	0.9792
3	2.2973	2.2980	1.3692	1.3695
4	2.6904	2.6921	1.6103	1.6067
5	2.8225	2.8267	1.6447	1.6491
6	3.6533	3.6721	1.6704	1.6663
7	3.6977	3.7061	1.9128	1.8225
8	4.1776	4.1891	1.9936	1.9713
9	4.2394	4.2516	2.0200	2.0200

(b) Circumferential wave number = 1

Mode	$L = 2$		$L = 10$	
	Present	FEM	Present	FEM
1	1.5333	1.5343	0.2476	0.2476
2	1.6011	1.6011	0.5249	0.5252
3	2.1667	2.1671	1.8209	0.8202
4	2.3123	2.3133	1.0146	1.0126
5	2.5430	2.5452	1.2246	1.2132
6	2.6808	2.6836	1.2247	1.2162
7	3.4695	3.4715	1.4927	1.4698
8	3.6193	3.6228	1.5595	1.5165

(c) Circumferential wave number = 2

Mode	$L = 2$		$L = 10$	
	Present	FEM	Present	FEM
1	0.9229	0.9245	1.1200	1.1288
2	1.1629	1.1671	1.1278	1.1362
3	2.1067	2.1079	1.1708	1.1842
4	2.1507	2.1536	1.1905	1.2027
5	2.5391	2.5420	1.3028	1.3148
6	2.8456	2.8470	1.4839	1.4891
7	3.0380	3.0443	1.7158	1.6955
8	3.3142	3.3200	1.8293	1.8225
9	3.8100	3.8219	1.9873	1.9464
10	3.9078	3.9188	2.0020	1.9682

The hollow isotropic cylinder

The frequencies for the circumferential vibrations of a hollow isotropic cylinder are given in Table 7 for the shape parameters $L = 1$ and $L = 5$. Again, H represents the ratio of length to mean radius, where the mean radius is the average of the distance to the outer and inner thicknesses. For this example, the thickness/mean radius ratio is fixed at 1.4. The dimensionless frequency is taken as $\omega^* = \omega a/c$, where $c = \sqrt{(G/\rho)}$ is the shear wave velocity. Very good agreement is obtained with the finite element results.

Comparisons were also made with the work of Soldatos and Hadjigeorgios (1990), who used an iterative solution of the three-dimensional equations of elasticity to predict the exact frequencies of vibration for isotropic circular cylindrical shells. Table 8 shows the results for a shell with a thickness/mean radius ratio of 0.3, a length/mean radius ratio of

Table 8. Frequency parameter ω^* for isotropic circular cylindrical shells

n	Mode 1		Mode 2		Mode 3	
	Present	Soldatos	Present	Soldatos	Present	Soldatos
1	1.33731	1.33761	2.37760	2.37781	3.93351	3.93343
2	1.32335	1.32371	2.72152	2.72196	4.42453	4.42468
3	1.52768	1.52805	3.16100	3.16159	5.11173	5.11234
4	1.92325	1.92695	3.67049	3.67122	5.90205	5.90307

1.0, a shear modulus of 1.0 and a Poisson ratio of 0.3. The values are given in terms of the frequency parameter $\omega^* = 2L_z\omega/\sqrt{2}$. The agreement is excellent.

The layered isotropic sphere

The layered sphere is composed of dissimilar isotropic materials in the radial direction, such as those encountered when modeling the strata of the earth. These geometries can be analysed by dividing the regions of integration between the different layers and substituting in the appropriate material properties. A two-layer sphere was considered with the interface located at $L_r/2$, where L_r is the outer radius of the sphere. For simplicity, the elastic constants of the inner core are fixed, with the Young modulus given as 2.5 and the Poisson ratio of 0.25. The Young modulus of the outer layer is then varied within a factor range of 0.1–10.0. The range of values of the first four frequencies are shown in Fig. 2, and are plotted versus the parameter β, which represents the log of the ratio between the two moduli.

The inhomogeneous sphere

The element coefficient matrices can be evaluated numerically, and it is therefore a simple procedure to determine the natural frequencies for solids with properties with any spatial variation. This can be used to represent the change in elastic constants or material density due to environmental causes such as temperature or manufacturing processes. For purposes of demonstration, a sphere with a simple linear variation in the Young modulus is considered, described by the equation

$$E(r) = E_0\left(1+\alpha\frac{r}{L_r}\right). \tag{19}$$

The approximate values are computed using the formulation. Figure 3 shows the first four dimensionless nonzero frequencies as a function of the parameter α.

The thick, orthotropic shell

Most studies of the free vibrations of orthotropic, thick shells have been used on approximate shell theories rather than solutions to the equations of elasticity. Such theories frequently make a significant number of assumptions regarding the type of deformation that can occur in the shell, and can often require restrictions regarding the effects of transverse normal stress, rotatory inertia, shear deformations and deformation of the normals. The present approach offers a useful alternative in that all effects are strictly enforced from an elasticity sense. Comparisons provide a useful means of evaluating the effects of assumptions typically used in anisotropic shell theories.

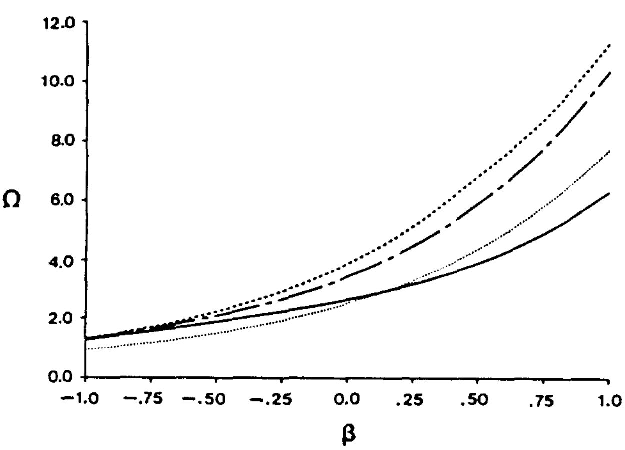

Fig. 2. Dimensionless frequencies for a layered sphere.

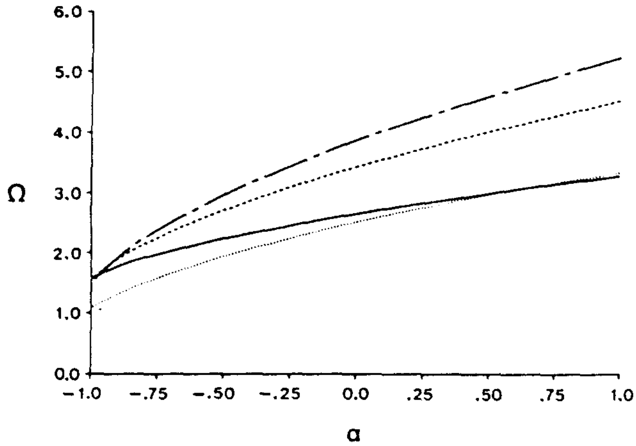

Fig. 3. Dimensionless frequencies for an inhomogeneous sphere.

The free vibrations of thick cylindrical shells are considered in this section for three different orthotropic materials: topaz, barytes and magnesium. The density and elastic constants used for these materials are those taken from Bergman (1938), Barrett (1952) and Bolef and De Klerk (1962). These values are given in Table 9. The resulting frequencies for the case of axisymmetric vibrations are given in Table 10. The frequency parameters ω^* are defined as the natural frequency divided by the shear wave velocity c. These values are taken as $c = \sqrt{(c_{55}/\rho)}$ for the topaz and barytes, and $c = \sqrt{(c_{44}/\rho)}$ for the magnesium. The frequency parameters are plotted as a function of mode number and the parameter δ, which is the ratio of shell thickness h to the wavelength L. The thickness/mean radius ratio was fixed at 1.0 for the geometries considered.

The frequency parameters are compared with the results of Mirsky (1964, 1965), who used both an anisotropic elasticity solution and an orthotropic shell theory. The results for the magnesium were extracted from a graph, while the results for the topaz and barytes were taken directly from a table. In general, the results are excellent.

The solid wood cylinder

The present method can also be applied to consider the vibrations of wood for purposes of evaluating elastic constants or their variation as a function of age, use, moisture content or other parameters. The density and elastic constants of Red Beech at a moisture content of 11% are given in Table 11 (Bodig and Goodman, 1973). The natural frequencies of a cylindrical sample with radius 1.0 and varying height are shown in Table 12, where the results are given in Hertz. The symbol (2) denotes a repeated root.

The orthotropic cross-ply shell

A situation similar to the dissimilar isotropic sphere is the cylinder consisting of layers composed of two different materials. The level of complexity can be increased by allowing each layer to be anisotropic. A two-layer, cross-ply cylindrical shell composed of graphite-

Table 9. Elastic stiffnesses C_{ij} expressed in units of 10^6 g cm^{-2}

C_{ij}	Topaz	Barytes	Magnesium
C_{11}	2871	912	597
C_{22}	3561	800	597
C_{33}	3005	1076	623
C_{44}	1100	121	169
C_{55}	1357	293	169
C_{66}	1330	283	171
C_{12}	1284	471	255
C_{13}	864	275	212
C_{23}	273	900	212

Table 10. Frequency parameter ω^* for thick, orthotropic cylindrical shells

(a) The Topaz shell: axisymmetric vibrations

δ	Mode							
	1		2		3		4	
	Present	Mirsky	Present	Mirsky	Present	Mirsky	Present	Mirsky
0	0.0000	0.00	1.6532	1.65	3.2712	3.27	4.9547	4.97
0.01	0.0879	0.088	1.6528	1.65	3.2731	3.27	4.9540	4.97
0.1	0.8695	0.87	1.6337	1.63	3.4449	3.44	4.8940	4.90
0.2	1.5040	1.50	1.8940	1.89	3.8678	3.87	4.7776	4.78
0.3	1.7548	1.75	2.6983	2.70	4.3965	4.40	4.7173	4.72
0.4	2.1250	2.12	3.4980	3.50	4.7144	4.72	5.0438	5.05
0.5	2.6004	2.60	4.1112	4.12	5.0587	5.06	5.6297	5.63

(b) The Barytes shell: axisymmetric vibrations

δ	Mode							
	1		2		3		4	
	Present	Mirsky	Present	Mirsky	Present	Mirsky	Present	Mirsky
0	0.0000	0.00	1.5728	1.57	3.2712	3.27	5.8018	5.84
0.01	0.1139	0.114	1.5725	1.58	3.2739	3.27	5.7998	5.84
0.1	1.1119	1.10	1.5777	1.57	3.5209	3.52	5.6672	5.70
0.2	1.5540	1.53	2.3443	2.36	4.1211	4.13	5.5104	5.53
0.3	1.8479	1.83	3.4434	3.47	4.8901	4.91	5.4751	5.51
0.4	2.2909	2.27	4.4336	4.45	5.6602	5.69	5.7397	5.76
0.5	2.8155	2.77	5.0691	5.08	6.2983	6.34	6.6155	6.64

(c) The Magnesium shell: axisymmetric vibrations

δ	Mode											
	1		2		3		4		5		6	
	Present	[36]	Present	[36]	Present	[36]	Present	[36]	Present	[36]	Present	[36]
0	0.0000	0.00	0.0000	0.00	1.8966	1.90	3.2909	3.29	3.7363	3.79	6.3092	6.32
0.05	0.3142	0.32	0.5474	0.57	1.8869	1.87	3.3583	3.37	3.7492	3.79	6.1771	6.18
0.10	0.6283	0.63	1.0775	1.08	1.8863	1.89	3.5429	3.54	3.7884	3.84	6.0368	6.00
0.15	0.9425	0.95	1.5103	1.47	1.9995	2.10	3.8105	3.81	3.8530	3.89	5.9225	5.89
0.20	1.2566	1.26	1.7202	1.72	2.3797	2.38	3.9417	4.00	4.1333	4.11	5.8441	5.80
0.25	1.5708	1.57	1.8530	1.85	2.8838	2.88	4.0528	4.05	4.4931	4.47	5.7992	5.77
0.30	1.8849	1.88	2.0093	2.00	3.4021	3.40	4.1846	4.21	4.8786	4.88	5.7955	5.79
0.35	2.1991	2.21	2.2000	2.21	3.9044	3.90	4.3352	4.38	5.2818	5.26	5.8421	5.84
0.40	2.4198	2.42	2.5133	2.52	4.3677	4.37	4.5027	4.58	5.6970	5.69	5.9557	6.00
0.45	2.6619	2.66	2.8274	2.84	4.6853	4.69	4.7643	4.76	6.1191	6.11	6.1604	6.21
0.50	2.9203	2.92	3.1416	3.16	4.8813	4.88	5.0751	5.07	6.4651	6.47	6.5510	6.55

Table 11. Elastic properties of Red Beech

ρ in g cm^{-3}	Modulus of elasticity and rigidity in 10^6 psi					
ρ	(E_L)	(E_R)	(E_T)	(G_{LR})	(G_{LT})	(G_{RT})
0.75	1.987	0.325	0.165	0.234	0.154	0.067
	Poisson ratios					
	(ν_{LR})	(ν_{LT})	(ν_{RT})	(ν_{TR})	(ν_{RL})	(ν_{TL})
	0.45	0.51	0.75	0.36	0.075	0.044

Table 12. Natural frequencies (in Hertz) of Red Beech cylinder as a function of height

Mode	$L = 2$	$L = 4$	$L = 6$	$L = 8$	$L = 10$
1	82.88 (2)	104.48 (2)	75.52	56.64	45.31
2	117.22 (2)	113.28	111.03 (2)	93.64 (2)	69.31 (2)
3	127.92 (2)	117.23 (2)	117.23 (2)	113.29	90.63
4	151.70 (2)	127.82 (2)	126.43 (2)	113.62 (2)	110.74 (2)
5	157.47	166.37 (2)	136.69 (2)	117.23 (2)	114.87 (2)
6	180.95 (2)	173.55 (2)	143.17 (2)	128.21 (2)	117.23 (2)
7	200.40 (2)	173.89 (2)	150.73 (2)	128.30 (2)	126.41 (2)
8	212.64	180.94 (2)	151.05	137.48 (2)	128.02 (2)
9	218.03	191.68	162.99 (2)	141.31 (2)	136.60 (2)
10	218.86	199.33 (2)	178.30 (2)	172.21	137.77

Table 13. Frequency parameter for a layered composite shell

m	n	$L/a = 2$		$L/a = 4$	
		Present	Kumar	Present	Kumar
1	0	1.721	1.611	3.441	3.234
1	1	1.177	1.133	2.285	2.178
1	2	0.842	1.477	1.403	—

epoxy was considered with the properties $E_1 = 30.0 \times 10^6$ psi, $E_2 = 0.75 \times 10^6$ psi, $\nu_{12} = 0.25$, $G_{12} = 0.375 \times 10^6$ psi and $G_{23}/E_2 = 0.416$. The thickness/mean radius ratio was taken as 0.2. The shell in this example has a [0/90] stacking sequence, which implies that the fibers in the outer layer run in the circumferential direction, while the fibers in the inner layer run in the axial direction. The results are compared with the finite element results of Kumar and Rao (1988), which were computed using a degenerated eight-noded isoparametric shell element with a shear coefficient of 5/6. The frequencies are tabulated in terms of the frequency parameter ω^*, defined as

$$\omega^* = \omega \frac{l^2}{\pi^2} \sqrt{\frac{\rho}{D_{11}}}, \tag{20}$$

where ρ is the material density, ω is the natural frequency, l is the cylinder length, and

$$D_{11} = \frac{\left(1 + \frac{E_1}{E_2}\right)}{2} Q_{11} \frac{t^3}{12}. \tag{21}$$

Here, t is the shell thickness and Q_{11} is the reduced stiffness in the direction of the fibers (Jones, 1975).

Results are shown in Table 13 as a function of m, the number of waves in the axial direction, n, the number of waves in the circumferential direction, and the ratio of the mean radius a to the shell length l. The results were based on a two-dimensional stress state and did not account for out-of-plane stiffness terms (i.e. $C_{rr} = C_{r\theta} = C_{rz} = 0$). Fairly good agreement is found between the two approaches.

5. SUMMARY

In this paper, approximate solutions of the three-dimensional equations of motion are developed for the frequencies of free vibrations of solid and hollow anisotropic cylinders and spheres. Several formulations of the Ritz method were applied to take advantage of the form of the constitutive relations and the geometry of the solid shape. Approximating functions involving combinations of power series, Fourier series and spherical harmonics were applied to a number of sample geometries of both isotropic and anisotropic constitution. Although many terms can be and were evaluated in closed form, it was sometimes advantageous to evaluate the coefficient matrices numerically to account for spatial variations in the elastic constants.

The present approach extends and generalizes earlier applications of this type of solution and is particularly useful for solids in which the principal material directions do not align with rectangular Cartesian coordinates. Numerous representative examples were considered, with very good accuracy being obtained in those cases for which comparisons were available.

Acknowledgement—This work was supported by the National Science Foundation under Grant Number MSS-9010099.

REFERENCES

Armenakas, A. E., Gazis, D. C. and Herrmann, G. (1969). *Free Vibrations of Circular Cylindical Shells*. Pergamon Press, Oxford.

Barrett, C. S. (1952). *Structure of Metals* (2nd Edn). McGraw-Hill, New York.

Bancroft, D. (1941). The velocity of longitudinal waves in cylindrical bars. *Phys. Rev.* **59**, 588–593.

Bergman, L. (1938). *Ultrasonics* (translated by H. Stafford Hatfield). G. Bell and Sons Ltd, London.

Bodig, J. and Goodman, J. R. (1973). Prediction of elastic parameters of wood. *Wood Sci.* **5**, 249–264.

Bolef, D. I. and De Klerk, J. (1962). Elastic constants of single-crystal Mo and W between 77° and 500° K. *J. Appl. Phys.* **33**, 2311–2314.

Chree, C. (1886). Longitudinal vibrations of a circular bar. *Q. J. Math.* **21**, 287–298.

Chree, C. (1889). The equations of an isotropic elastic solid in polar and cylindrical coordinates, their solution and application. *Trans. Cambr. Phil. Soc.* **14**, 250–369.

Demarest, H. H. (1969). Cube-resonance method to determine the elastic constants of solids. *J. Acous. Soc. Am.* **49**, 768–775.

Eer Nisse, E. P. (1967). Variational method for electroelastic vibration analysis. *IEEE Trans. Sonics Ultrason.* **SU-14**, 153–160.

Fung, Y. C. (1965). *Foundations of Solid Mechanics*. Prentice-Hall, Englewood Cliffs, NJ.

Gladwell, G. M. L. and Tahbildar, U. C. (1972). Finite element analysis of the axisymmetric vibrations of cylinders. *J. Sound Vibr.* **22**, 143–157.

Gladwell, G. M. L. and Vijay, D. K. (1975). Natural frequencies of free finite-length circular cylinders. *J. Sound Vibr.* **42**, 387–397.

Heyliger, P. R. (1991). Axisymmetric free vibrations of finite anisotropic cylinders. *J. Sound Vibr.* **148**, 507–520.

Heyliger, P. R., Ledbetter, H. and Austin, M. (1989). Resonating orthotropic-cube method for elastic constants. *Dyn. Elastic Modulus Measure. Mater.* **ASTM STP 1045**.

Holland, R. (1968). Resonant properties of piezoelectric ceramic rectangular parallelepipeds. *J. Acoust. Soc. Am.* **43**, 988–997.

Hosseini-Hashemi, S. (1986). E = the sound and vibration resulting from the impact of spheres. Ph.D. Thesis, The City University of London, London.

Hosseini-Hashemi, S. and Anderson, J. S. (1988). Orthogonality and normalization of torsional modes of vibration of solid elastic spheres. *J. Sound Vibr.* **121**, 511–524.

Hutchinson, J. R. (1972). Axisymmetric vibrations of a free finite-length rod. *J. Acoust. Soc. Am.* **51**, 233–240.

Jones, R. M. (1975). *Mech. Comp. Mater.* McGraw-Hill, New York.

Kelvin, Lord (1863). On the rigidity of the earth. *Phil. Trans. R. Soc.* **153**, 573–597.

Kumar, R. R. and Rao, Y. V. K. (1988). Free vibrations of multilayered thick composite shells. *Comput. Struct.* **28**, 717–722.

Lamb, H. (1882). On the vibrations of an elastic sphere. *Proc. Lond. Math. Soc.* **13**, 189–212.

Landau, L. D., Lifshitz, A. M., Kosevich, A. M. and Pitaevskii, L. P. (1986). *Theory of Elasticity*. Pergamon Press, New York.

Lapwood, E. R. and Usami, T. (1961). *Free Oscillations of the Earth*. Cambridge University Press, Cambridge.

Lusher, C. P. and Hardy, W. N. (1988). Axisymmetric free vibrations of a transversely isotropic finite cylindrical rod. *ASME J. Appl. Mech.* **55**, 855–862.

McMahon, G. W. (1970). Finite difference analysis of the vibrations of solid cylinders. *J. Acoust. Soc. Am.* **48**, 307–312.

McMahon, G. W. (1964). Experimental study of the vibrations of solid, isotropic, elastic cylinders. *J. Acoust. Soc. Am.* **36**, 85–92.

McNiven, H. D. and Perry, D. C. (1962). Axially symmetric waves in finite, elastic rods. *J. Acoust. Soc. Am.* **34**, 433–437.

Migliori, A., Visscher, W. M., Brown, S. E., Fisk, Z., Cheong, S.-W., Alten, B., Ahrens, E. T., Martin, L. A., Maynard, J. D., Huang, Y., Kirk, D. R., Gillis, K. A., Kim, H. K. and Chan, M. H. W. (1990). *Phys. Rev.* **B41**, 2098–3002.

Mindlin, R. D. and McNiven, H. D. (1960). Axially symmetric waves in elastic rods. *ASME J. Appl. Mech.* **27**, 145–151.

Mirsky, I. (1964). Vibrations of orthotropic, thick, cylindrical shells. *J. Acoust. Soc. Am.* **36**, 41–51.

Mirsky, I. (1965). Three-dimensional and shell-theory analysis of axisymmetric vibrations of orthotropic shells. *J. Acoust. Soc. Am.* **39**, 549–555.

Mochizuki, E. (1988). Sphere-resonance method to determine elastic constants of crystal. *J. Appl. Phys.* **63**, 5668–5673.

Morse, R. W. (1954). Compressional waves along an anisotropic circular cylinder having hexagonal symmetry. *J. Acoust. Soc. Am.* **26**, 1018–1021.

Ness, N. F., Harrison, J. C. and Slichter, L. B. (1961). Observation of the free oscillations of the earth. *J. Geophys. Res.* **66**, 621–629.

Ohno, I. (1976). Free vibration of a rectangular parallelepiped crystal and its application to determination of elastic constants of orthorhombic crystals. *J. Phys. Earth* **24**, 355–379.

Pekeris, C. L., Alterman, Z. and Jarosch, H. (1961). Comparison of theoretical with observed values of the periods of free oscillations of the earth. *Proc. Nat. Acad. Sci.* **47**, 91–98.

Pochhammer, L. (1876). Ueber die Fortpflanzungsgeschwindigkeiten Schwingungen in einem unbergranzten isotropen Kreiscylinder. *Zeitesche Mathematique* **81**, 324–336.

Reddy, J. N. (1984). *Energy and Variational Methods in Applied Mechanics*. Wiley, New York.

Sato, Y. and Usami, T. (1962a). Basic study on the oscillation of a homogeneous elastic sphere: Part I, frequency of the free oscillations. *Geophys. Mag.* **31**, 15–24.

Sato, Y. and Usami, T. (1962b). Basic study on the oscillation of a homogeneous elastic sphere: Part II, distribution of displacement. *Geophys. Mag.* **31**, 25–47.

Sokolnikoff, I. S. (1956). *Mathematical Theory of Elasticity*. McGraw-Hill, New York.

Soldatos, K. P. and Hadjigeorgios, V. P. (1990). Three-dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels. *J. Sound Vibr.* **137**, 369–384.
Usami, T. (1958). Elastic waves in a homogeneous and isotropic medium generated from a spherical origin. *Geophys. Mag.* **29**, 1–9.
Usami, T. and Sato, Y. (1962). Torsional oscillation of a homogeneous elastic spheroid. *Bull. Seis. Soc. Am.* **52**, 469–484.
Visscher, W. M., Migliori, A., Bell, T. M. and Reinert, R. A. (1991). On the normal modes of free vibration of inhomogeneous and anisotropic elastic objects. *J. Acoust. Soc. Am.* **90**, 2154–2162.

APPENDIX A: EQUATIONS OF MOTION AND COEFFICIENT MATRICES

Cylindrical coordinates

The equations of motion for the orthotropic cylinder are given by

$$C_{11}\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}\right)+\left(\frac{1}{r}C_{12}+\frac{1}{r}C_{66}\right)\frac{\partial^2 V}{\partial r\,\partial\theta}+(C_{13}+C_{55})\frac{\partial^2 W}{\partial r\,\partial z}+\left(\frac{1}{r}C_{13}-\frac{1}{r}C_{23}\right)\frac{\partial W}{\partial z}$$
$$-\left(\frac{1}{r^2}C_{22}+\frac{1}{r^2}C_{66}\right)\frac{\partial V}{\partial\theta}-\frac{1}{r^2}C_{22}U+C_{55}\frac{\partial^2 U}{\partial z^2}+\frac{1}{r^2}C_{66}\frac{\partial^2 U}{\partial\theta^2}=\rho\ddot{U}. \quad \text{(A1)}$$

$$\left(\frac{1}{r}C_{12}+\frac{1}{r}C_{66}\right)\frac{\partial^2 U}{\partial r\,\partial\theta}+\left(\frac{1}{r}C_{23}+\frac{1}{r}C_{44}\right)\frac{\partial^2 W}{\partial\theta\,\partial z}+\frac{1}{r^2}C_{22}\frac{\partial^2 V}{\partial\theta^2}+\left(\frac{1}{r^2}C_{22}+\frac{1}{r^2}C_{66}\right)\frac{\partial U}{\partial\theta}$$
$$+C_{44}\frac{\partial^2 V}{\partial z^2}+C_{66}\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}C_{66}\frac{\partial V}{\partial r}-\frac{1}{r^2}C_{66}V=\rho\ddot{V}. \quad \text{(A2)}$$

$$(C_{13}+C_{55})\frac{\partial^2 U}{\partial r\,\partial z}+\left(\frac{1}{r}C_{23}+\frac{1}{r}C_{44}\right)\frac{\partial^2 V}{\partial\theta\,\partial z}+\left(\frac{1}{r}C_{23}+\frac{1}{r}C_{55}\right)\frac{\partial U}{\partial z}+C_{33}\frac{\partial^2 W}{\partial z^2}+\frac{1}{r^2}C_{44}\frac{\partial^2 W}{\partial\theta^2}$$
$$+C_{55}\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}C_{55}\frac{\partial W}{\partial r}=\rho\ddot{W}. \quad \text{(A3)}$$

The submatrices for the Ritz coefficients corresponding to this formulation are written as

$$K^{11}_{ij}=\int_V\left[\frac{C_{11}}{L_r^2}\frac{\partial\phi_i^u}{\partial r}\frac{\partial\phi_j^u}{\partial r}+\frac{1}{r}\frac{C_{12}}{L_r^2}\left(\frac{\partial\phi_i^u}{\partial r}\phi_j^u+\phi_i^u\frac{\partial\phi_j}{\partial r}\right)+\frac{1}{r^2}\frac{C_{22}}{L_r^2}\phi_i^u\phi_j^u+\frac{C_{55}}{L_r^2}\frac{\partial\phi_i^u}{\partial z}\frac{\partial\phi_j^u}{\partial z}+\frac{1}{r^2}\frac{C_{66}}{L_r^2}\frac{\partial\phi_i^u}{\partial\theta}\frac{\partial\phi_j^u}{\partial\theta}\right]\mathrm{d}V, \quad \text{(A4)}$$

$$K^{12}_{ij}=K^{21}_{ji}=\int_V\left[\frac{1}{r}\frac{C_{12}}{L_r^2}\frac{\partial\phi_i^u}{\partial r}\frac{\partial\phi_j^v}{\partial\theta}+\frac{1}{r^2}\frac{C_{22}}{L_r^2}\phi_i^u\frac{\partial\phi_j^v}{\partial\theta}-\frac{1}{r}\frac{C_{66}}{L_r^2}\frac{\partial\phi_i^u}{\partial\theta}\left(\frac{\partial\phi_j^v}{\partial r}-\frac{\phi_j^v}{r}\right)\right]dV, \quad \text{(A5)}$$

$$K^{13}_{ij}=K^{31}_{ji}=\int_V\left[\frac{C_{13}}{L_rL_z}\frac{\partial\phi_i^u}{\partial r}\frac{\partial\phi_j^w}{\partial z}+\frac{1}{r}\frac{C_{23}}{L_rL_z}\phi_i^u\frac{\partial\phi_j^w}{\partial z}+\frac{C_{55}}{L_rL_z}\frac{\partial\phi_i^u}{\partial z}\frac{\partial\phi_j^w}{\partial r}\right]\mathrm{d}V, \quad \text{(A6)}$$

$$K^{22}_{ij}=\int_V\left[\frac{1}{r^2}\frac{C_{22}}{L_r^2}\frac{\partial\phi_i^v}{\partial\theta}\frac{\partial\phi_j^v}{\partial\theta}+\frac{C_{44}}{L_r^2}\frac{\partial\phi_i^v}{\partial z}\frac{\partial\phi_j^v}{\partial z}+\frac{C_{66}}{L_r^2}\left(\frac{\partial\phi_i^v}{\partial r}-\frac{\phi_i^v}{r}\right)\left(\frac{\partial\phi_j^v}{\partial r}-\frac{\phi_j^v}{r}\right)\right]\mathrm{d}V, \quad \text{(A7)}$$

$$K^{23}_{ij}=K^{32}_{ji}=\int_V\left[\frac{1}{r}\frac{C_{23}}{L_rL_z}\frac{\partial\phi_i^v}{\partial\theta}\frac{\partial\phi_j^w}{\partial z}+\frac{1}{r}\frac{C_{44}}{L_rL_z}\frac{\partial\phi_i^v}{\partial z}\frac{\partial\phi_j^w}{\partial\theta}\right]\mathrm{d}V, \quad \text{(A8)}$$

$$K^{33}_{ij}=\int_V\left[\frac{C_{33}}{L_r^2}\frac{\partial\phi_i^w}{\partial z}\frac{\partial\phi_j^w}{\partial z}+\frac{1}{r^2}\frac{C_{44}}{L_r^2}\frac{\partial\phi_i^w}{\partial\theta}\frac{\partial\phi_j^w}{\partial\theta}+\frac{C_{55}}{L_r^2}\frac{\partial\phi_i^w}{\partial r}\frac{\partial\phi_j^w}{\partial r}\right]\mathrm{d}V. \quad \text{(A9)}$$

$$M_{ij}^{11} = M_{ij}^{22} = M_{ij}^{33} = \int_V \psi_i \psi_j r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z. \tag{A10}$$

Spherical coordinates

The equations of motion for the orthotropic sphere are given by

$$\begin{aligned}
&C_{11}\left(\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r}\right) + C_{12}\left(\frac{1}{r}\frac{\partial^2 V}{\partial r \partial \phi} + \frac{1}{r^2}\frac{\partial V}{\partial \phi} + \frac{U}{r^2}\right)\\
&C_{13}\left(\frac{1}{r \sin\phi}\frac{\partial^2 W}{\partial r \partial \theta} + \frac{1}{r^2 \sin\phi}\frac{\partial W}{\partial \theta} + \frac{U}{r^2} + \frac{\cot\phi}{r}\frac{\partial V}{\partial r} + \frac{\cot\phi}{r^2}V\right) - C_{22}\left(\frac{1}{r^2}\frac{\partial V}{\partial \phi} + \frac{U}{r^2}\right)\\
&C_{23}\left(\frac{1}{r^2 \sin\phi}\frac{\partial W}{\partial \theta} + \frac{\cot\phi}{r^2}V + \frac{1}{r^2}\frac{\partial V}{\partial \phi} + \frac{2U}{r^2}\right) - C_{33}\left(\frac{1}{r^2 \sin\phi}\frac{\partial W}{\partial \theta} + \frac{U}{r^2} + \frac{\cot\phi}{r^2}V\right)\\
&C_{44}\left(\frac{1}{r^2 \sin^2\phi}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r \sin\phi}\frac{\partial^2 W}{\partial r \partial \theta} - \frac{1}{r^2 \sin\phi}\frac{\partial W}{\partial \theta}\right)\\
&C_{66}\left(\frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial U}{\partial \phi} + \frac{1}{r}\frac{\partial^2 V}{\partial r \partial \phi} + \frac{\cot\phi}{r}\frac{\partial V}{\partial r} - \frac{1}{r^2}\frac{\partial V}{\partial \phi} - \frac{\cot\phi}{r^2}V\right) = \rho \ddot{U}_r.
\end{aligned} \tag{A11}$$

$$\begin{aligned}
&C_{12}\left(\frac{1}{r}\frac{\partial^2 U}{\partial \phi \partial r} + \frac{\cot\phi}{r}\frac{\partial U}{\partial r}\right) + C_{22}\left(\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial V}{\partial \phi} + \frac{1}{r^2}\frac{\partial U}{\partial \phi} + \frac{\cot\phi}{r^2}U\right)\\
&C_{23}\left(\frac{1}{r^2 \sin\phi}\frac{\partial^2 W}{\partial \theta \partial \phi} + \frac{1}{r^2}\frac{\partial U}{\partial \phi} - \frac{V}{r^2}\right) - C_{13}\left(\frac{\cot\phi}{r}\frac{\partial U}{\partial r}\right) - C_{33}\left(\frac{\cot\phi}{r^2 \sin\phi}\frac{\partial W}{\partial \theta} + \frac{\cot\phi}{r^2}U + \frac{\cot^2\phi}{r^2}V\right)\\
&C_{66}\left(\frac{1}{r}\frac{\partial^2 U}{\partial \phi \partial r} + \frac{2}{r^2}\frac{\partial U}{\partial \phi} + \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r}\right) + C_{55}\left(\frac{1}{r^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2 \sin\phi}\frac{\partial^2 W}{\partial \theta \partial \phi} - \frac{\cot\phi}{r^2 \sin\phi}\frac{\partial W}{\partial \theta}\right) = \rho \ddot{U}_\phi.
\end{aligned} \tag{A12}$$

$$\begin{aligned}
&C_{13}\left(\frac{1}{r \sin\phi}\frac{\partial^2 U}{\partial r \partial \theta}\right) + C_{23}\left(\frac{1}{r^2 \sin\phi}\right)\left(\frac{\partial^2 V}{\partial \phi \partial \theta} + \frac{\partial U}{\partial \theta}\right) + C_{33}\left(\frac{1}{r^2 \sin^2\phi}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r^2 \sin\phi}\frac{\partial U}{\partial \theta} + \frac{\cos\phi}{r^2 \sin^2\phi}\frac{\partial V}{\partial \theta}\right)\\
&+ C_{55}\left(\frac{1}{r^2 \sin\phi}\frac{\partial^2 V}{\partial \phi \partial \theta} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial W}{\partial \phi} + \frac{W}{r^2} + \frac{\cos\phi}{r^2 \sin^2\phi}\frac{\partial V}{\partial \theta} - \frac{\cot^2\phi}{r^2}W\right)\\
&+ C_{44}\left(\frac{1}{r \sin\phi}\frac{\partial^2 U}{\partial r \partial \theta} + \frac{2}{r^2 \sin\phi}\frac{\partial U}{\partial \theta} + \frac{\partial^2 W}{\partial r^2} + \frac{2}{r}\frac{\partial W}{\partial r} - \frac{2}{r^2}W\right) = \rho \ddot{U}_\theta.
\end{aligned} \tag{A13}$$

The elements of the coefficient matrices are expressed as

$$M_{ij}^{11} = M_{ij}^{22} = M_{ij}^{33} = \int_V \psi_i \psi_j r^2 \sin\phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{A14}$$

$$\begin{aligned}
K_{ij}^{11} = \int_V \Bigg(& C_{11}\frac{\partial \psi_i^u}{\partial r}\frac{\partial \psi_j^u}{\partial r} + \frac{C_{12}}{r}\frac{\partial \psi_i^u}{\partial r}\psi_j^u + \frac{C_{13}}{r}\frac{\partial \psi_i^u}{\partial r}\psi_j + \frac{C_{12}}{r}\frac{\partial \psi_j^u}{\partial r}\psi_i^u + \frac{C_{22}}{r^2}\psi_i^u\psi_j^u + \frac{2C_{23}}{r^2}\psi_i^u\psi_j^u + \frac{C_{13}}{r}\frac{\partial \psi_j^u}{\partial r}\psi_i^u\\
& + \frac{C_{33}}{r^2}\psi_i^u\psi_j^u + \frac{C_{66}}{r^2}\frac{\partial \psi_i^u}{\partial \phi}\frac{\partial \psi_j^u}{\partial \phi} + \frac{C_{44}}{r^2 \sin^2\phi}\frac{\partial \psi_i^u}{\partial \theta}\frac{\partial \psi_j^u}{\partial \theta} + \frac{C_{44}}{r^2 \sin^2\phi}\frac{\partial \psi_i^u}{\partial \theta}\frac{\partial \psi_j^u}{\partial \theta}\Bigg) r^2 \sin\phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi.
\end{aligned} \tag{A15}$$

$$\begin{aligned}
K_{ij}^{12} = K_{ji}^{21} = \int_V \Bigg[& \frac{C_{12}}{r}\frac{\partial \psi_i^u}{\partial r}\frac{\partial \psi_j^v}{\partial \phi} + \frac{C_{13}}{r}\cot\phi\frac{\partial \psi_i^u}{\partial r}\psi_j^v + \frac{C_{22}}{r^2}\frac{\partial \psi_j^v}{\partial \phi}\psi_i^u + \frac{C_{23}}{r^2}\cot\phi\,\psi_i^u\psi_j^v + \frac{C_{23}}{r^2}\frac{\partial \psi_j^v}{\partial \phi}\psi_i^u\\
& + \frac{C_{33}}{r^2}\cot\phi\,\psi_i^u\psi_j^v + \frac{C_{66}}{r}\frac{\partial \psi_i^u}{\partial \phi}\left(\frac{\partial \psi_j^v}{\partial r} - \frac{\psi_j^v}{r}\right)\Bigg] r^2 \sin\phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi.
\end{aligned} \tag{A16}$$

$$K_{ij}^{13} = K_{ji}^{31} = \int_V \left[\frac{C_{13}}{r \sin\phi}\frac{\partial \psi_j^w}{\partial \theta} + \frac{C_{23}}{r^2 \sin\phi}\frac{\partial \psi_j^w}{\partial \theta}\psi_i^u + \frac{C_{33}}{r^2 \sin\phi}\frac{\partial \psi_j^w}{\partial \theta}\psi_i^u + \frac{C_{44}}{r \sin\phi}\frac{\partial \psi_i^u}{\partial \theta}\left(\frac{\partial \psi_j^w}{\partial r} - \frac{\psi_j^w}{r}\right)\right] r^2 \sin\phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{A17}$$

$$\begin{aligned}
K_{ij}^{22} = \int_V \Bigg[& \frac{C_{22}}{r^2}\frac{\partial \psi_i^v}{\partial \phi}\frac{\partial \psi_j^v}{\partial \theta} + \frac{C_{23}}{r^2}\cot\phi\frac{\partial \psi_i^v}{\partial \theta}\psi_j^v + \frac{C_{23}}{r^2}\cot\phi\,\psi_i^v\frac{\partial \psi_j^v}{\partial \theta} + \frac{C_{33}}{r^2}\cot^2\phi\,\psi_i^v\psi_j^v\\
& + C_{66}\left(\frac{\partial \psi_i^v}{\partial r} - \frac{\psi_i^v}{r}\right)\left(\frac{\partial \psi_j^v}{\partial r} - \frac{\psi_j^v}{r}\right) + \frac{C_{55}}{r^2 \sin^2\phi}\frac{\partial \psi_i^v}{\partial \theta}\frac{\partial \psi_j^v}{\partial \theta}\Bigg] r^2 \sin\phi \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi.
\end{aligned} \tag{A18}$$

$$K_{ij}^{23} = K_{ji}^{32} = \int_V \left[\frac{C_{23}}{r^2 \sin\phi} \frac{\partial \psi_i^\theta}{\partial \phi} \frac{\partial \psi_j^r}{\partial \theta} + \frac{C_{13}}{r^2 \sin\phi} \cot\phi \frac{\partial \psi_j^r}{\partial \theta} \psi_i^\theta + \frac{C_{55}}{r \sin\phi} \frac{\partial \psi_i^\theta}{\partial \theta} \times \left(\frac{1}{r} \frac{\partial \psi_j^r}{\partial \phi} - \frac{\cot\phi}{r} \psi_j^r \right) \right] r^2 \sin\phi \, dr \, d\theta \, d\phi. \quad (A19)$$

$$K_{ij}^{33} = \int_V \left[\frac{C_{33}}{r^2 \sin^2\phi} \frac{\partial \psi_i^r}{\partial \theta} \frac{\partial \psi_j^r}{\partial \theta} + C_{55} \left(\frac{1}{r} \frac{\partial \psi_i^r}{\partial \phi} - \frac{\cot\phi}{r} \psi_i^r \right) \left(\frac{1}{r} \frac{\partial \psi_j^r}{\partial \phi} - \frac{\cot\phi}{r} \psi_j^r \right) C_{44} \left(\frac{\partial \psi_i^r}{\partial r} - \frac{\psi_i^r}{r} \right) \right] r^2 \sin\phi \, dr \, d\theta \, d\phi. \quad (A20)$$

Rectangular Cartesian coordinates

The elements of the coefficient matrices are given by

$$K_{ij}^{11} = \int_V \left(C_{11} \frac{\partial \phi_i^u}{\partial x} \frac{\partial \phi_j^u}{\partial x} + C_{55} \frac{\partial \phi_i^u}{\partial z} \frac{\partial \phi_j^u}{\partial z} + C_{66} \frac{\partial \phi_i^u}{\partial y} \frac{\partial \phi_j^u}{\partial y} \right) dV, \quad (A21)$$

$$K_{ij}^{12} = \int_V \left(C_{12} \frac{\partial \phi_i^u}{\partial x} \frac{\partial \phi_j^v}{\partial y} + C_{66} \frac{\partial \phi_i^u}{\partial y} \frac{\partial \phi_j^v}{\partial x} \right) dV = K_{ji}^{21}, \quad (A22)$$

$$K_{ij}^{13} = \int_V \left(C_{13} \frac{\partial \phi_i^u}{\partial x} \frac{\partial \phi_j^w}{\partial z} + C_{55} \frac{\partial \phi_i^u}{\partial z} \frac{\partial \phi_j^w}{\partial x} \right) dV = K_{ji}^{31}, \quad (A23)$$

$$K_{ij}^{22} = \int_V \left(C_{22} \frac{\partial \phi_i^v}{\partial y} \frac{\partial \phi_j^v}{\partial y} + C_{44} \frac{\partial \phi_i^v}{\partial z} \frac{\partial \phi_j^v}{\partial z} + C_{66} \frac{\partial \phi_i^v}{\partial x} \frac{\partial \phi_j^v}{\partial x} \right) dV, \quad (A24)$$

$$K_{ij}^{23} = \int_V \left(C_{23} \frac{\partial \phi_i^v}{\partial y} \frac{\partial \phi_j^w}{\partial z} + C_{44} \frac{\partial \phi_i^v}{\partial z} \frac{\partial \phi_j^w}{\partial y} \right) dV = K_{ji}^{32}, \quad (A25)$$

$$K_{ij}^{33} = \int_V \left(C_{33} \frac{\partial \phi_i^w}{\partial z} \frac{\partial \phi_j^w}{\partial z} + C_{44} \frac{\partial \phi_i^w}{\partial y} \frac{\partial \phi_j^w}{\partial y} + C_{55} \frac{\partial \phi_i^w}{\partial x} \frac{\partial \phi_j^w}{\partial x} \right) dV. \quad (A26)$$

APPENDIX B: APPROXIMATION FUNCTIONS FOR SPHERE

Table B1. Example of approximation functions for a sphere

(a) Torsional modes

n	m	u_r	u_ϕ	u_θ
1	0	0	0	$\sin\phi$
1	1	0	$\cos\theta$	$\cos\phi \sin\theta$
			$\sin\theta$	$\cos\phi \cos\theta$
2	0	0	0	$\sin 2\theta$
2	1	0	$\cos\phi \cos\theta$	$\cos 2\phi \sin\theta$
			$\cos\phi \sin\theta$	$\cos 2\phi \cos\theta$
2	2	0	$\sin\phi \cos 2\theta$	$\sin 2\phi \sin 2\theta$
			$\sin\phi \sin 2\theta$	$\sin 2\phi \cos 2\theta$
3	0	0	0	$3 \sin\phi - 15 \cos^2\phi \sin\phi$
3	1	0	$(5 \cos^2\phi - 1) \cos\theta$	$\cos\phi (5 \cos^2\phi - 10 \sin^2\phi - 1) \sin\theta$
			$(5 \cos^2\phi - 1) \sin\theta$	$\cos\phi (5 \cos^2\phi - 10 \sin^2\phi - 1) \cos\theta$
3	2	0	$\sin 2\phi \cos 2\theta$	$(2 \cos^2\phi \sin\phi \sin^3\phi) \sin 2\theta$
			$\sin 2\phi \sin 2\theta$	$(2 \cos^2\phi \sin\phi \sin^3\phi) \cos 2\theta$
3	3	0	$\sin^2\phi \cos 3\theta$	$\sin^2\phi \cos\phi \sin 3\theta$
			$\sin^2\phi \sin 3\theta$	$\sin^2\phi \cos\phi \cos 3\theta$

(b) Spheroidal modes

n	m	u_r	u_ϕ	u_θ
0	0	1	0	0
1	0	$\cos\phi$	$\sin\phi$	0
1	1	$\sin\phi \sin\theta$	$\cos\phi \cos\theta$	$\sin\theta$
		$\sin\phi \cos\theta$	$\cos\phi \sin\theta$	$\cos\theta$
2	0	$3 \cos^2\phi - 1$	$\sin 2\phi$	0
2	1	$3 \sin\phi \cos\phi \sin\theta$	$\cos 2\phi \sin\theta$	$\cos\phi \sin\theta$
		$3 \sin\phi \cos\phi \cos\theta$	$\cos 2\phi \cos\theta$	$\cos\phi \cos\theta$
2	2	$3 \sin^2\phi \cos 2\theta$	$\sin 2\phi \cos 2\theta$	$\sin\phi \sin 2\theta$
		$3 \sin^2 \sin \cos 2\theta$	$\sin 2\phi \sin s\theta$	$\sin\phi \cos 2\theta$